THE UNIVERSITY OF ALBERTA

ROTATING DISKS IN GENERAL RELATIVITY

by

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A THESIS

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ABSTRACT

The object of this thesis is to determine and study the approximate gravitational field of an infinitismally thin rotating disk.

The metric is reviewed and the linearized gravitational field obtained. A survey is made of hypersurfaces in general relativity and the results are used to determine the energy tensor of the disk. An approach is outlined which enables one to determine the second order corrections to the field and energy tensor. The properties of the metric on the disk and in the far field limit are discussed. A short account of the gravitational red shift and the Lense-Thirring effect is given.

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CHAPTER I

The Linearized Gravitational Field of a Rotating Disk

I. Introduction.

The purpose of this dissertation is to examine the approximate gravitational field of a thin rotating disk of gravitating particles. In this section we shall outline the arguments which lead to the most general metric which determines a stationary axially symmetric gravitational field.

In general the line element has the form*

$$ds^2 = g_{\alpha\beta}\, dx^\alpha\, dx^\beta \quad . \tag{1.1.1}$$

The requirements of axial symmetry and time independence imply $g_{\alpha\beta} = g_{\alpha\beta}(x^1, x^2)$ where we have chosen $x^3 = \phi$, $x^4 = \tau$ to be the angular and time variables respectively. The gravitational field is invariant under the simultaneous transformation, $\phi \to -\phi$, $\tau \to -\tau$ since this leaves the angular velocity and thus the source of the gravitational field unchanged. This invariance of the metric requires

$$g_{13} = g_{14} = g_{23} = g_{24} = 0 \quad . \tag{1.1.2}$$

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* Greek indices are assumed to take the values 1 to 4, Latin indices 2 to 4.

Thus (1.1.1) reduces to

$$ds^2 = g_{11}(dx^1)^2 + g_{22}(dx^2)^2 + 2g_{12}\, dx^1\, dx^2 + 2g_{34}\, d\phi\, d\tau + g_{33}\, d\phi^2 + g_{44}\, d\tau^2 \quad . \tag{1.1.3}$$

The coordinates (x^1,x^2) can be chosen such that [1]

$$g_{11}(dx^1)^2 + 2g_{12}dx^1\, dx^2 + g_{22}\, (dx^2)^2 = A^2(x^1,x^2)((dx^1)^2 + (dx^2)^2) \ . \tag{1.1.4}$$

Hence if we now define

$$\begin{aligned} g_{33}(x^1,x^2) &= B^2(x^1,x^2) - C^2(x^1,x^2)\ \psi^2(x^1,x^2) \quad , \\ g_{34}(x^1,x^2) &= C^2(x^1,x^2)\ \psi(x^1,x^2) \quad , \\ g_{44}(x^1,x^2) &= -C^2(x^1,x^2) \end{aligned} \tag{1.1.5}$$

then equation (1.1.3) takes the form

$$ds^2 = A^2(x^1,x^2)((dx^1)^2 + (dx^2)^2) + B^2(x^1,x^2)\ d\phi^2 - C^2(x^1,x^2)(d\tau - \psi(x^1,x^2)d\phi)^2 \quad . \tag{1.1.6}$$

In order to obtain further simplification it is necessary to use the vacuum field equations. A variation of the argument used in the static case (see Synge [2]) leads to the result

$$B(\rho,z)C(\rho,z) = \rho \tag{1.1.7}$$

where we have put $x^1 = \rho$, $x^2 = z$. It is convenient to define

$$\begin{aligned} C(\rho,z) &= e^{\lambda} \quad , \\ B(\rho,z) &= \rho e^{-\lambda} \quad , \\ A(\rho,z) &= e^{\nu-\lambda} \end{aligned} \tag{1.1.8}$$

where ν and λ are functions of ρ and z . The final result is

$$ds^2 = e^{2(\nu-\lambda)}(d\rho^2+dz^2) + \rho^2 e^{-2\lambda}d\phi^2 - e^{2\lambda}(d\tau-\psi d\phi)^2 \tag{1.1.9}$$

where λ , ν , ψ are functions of ρ and z .

The vacuum field equations [3] (i.e. $R_{\alpha\beta} = 0$) take the form

$$\nu_\rho = \rho(\lambda_\rho^2 - \lambda_z^2) - \frac{1}{4}\frac{e^{4\lambda}}{\rho}(\psi_\rho^2 - \psi_z^2) \quad , \tag{1.1.10}$$

$$\nu_z = 2\rho\lambda_\rho\lambda_z - \frac{1}{2}\frac{e^{4\lambda}}{\rho}\psi_\rho\psi_z \quad , \tag{1.1.11}$$

$$\lambda_{\rho\rho} + \frac{1}{\rho}\lambda_\rho + \lambda_{zz} = -\frac{1}{2}\frac{e^{4\lambda}}{\rho^2}(\psi_\rho^2 + \psi_z^2) \ , \tag{1.1.12}$$

$$\psi_{\rho\rho} - \frac{1}{\rho}\psi_\rho + \psi_{zz} = -4\,(\lambda_\rho\psi_\rho + \lambda_z\psi_z) \quad . \tag{1.1.13}$$

Equation (1.1.13) can be rewritten as follows:

$$\frac{\partial}{\partial\rho}(\rho^{-1} e^{4\lambda}\psi_\rho) + \frac{\partial}{\partial z}(\rho^{-1} e^{4\lambda}\psi_z) = 0 \quad . \tag{1.1.14}$$

The form of the above equation suggests the introduction of a function Φ defined by

$$\begin{aligned} \Phi_z &= \rho^{-1} e^{4\lambda}\psi_\rho \quad , \\ \Phi_\rho &= -\rho^{-1} e^{4\lambda}\psi_z \quad . \end{aligned} \tag{1.1.15}$$

It has been demonstrated by Ernst [4] that the substitution

$$\mathcal{E} = e^{2\lambda} + i\,\Phi \tag{1.1.16}$$

implies that (1.1.12) and (1.1.13) are equivalent to the single complex equation

$$(\mathrm{Re}\,\mathcal{E})(\nabla^2\mathcal{E}) = (\nabla\mathcal{E})^2 \tag{1.1.17}$$

where ∇ is the gradient operator for a three dimensional flat space.

The imaginary part of (1.1.17) gives the following equation for Φ

$$\nabla^2\Phi = 4(\lambda_\rho\Phi_\rho + \lambda_z\Phi_z) \tag{1.1.18}$$

which replaces equation (1.1.13).

To obtain the linearized equations for λ , Φ , ν and ψ one assumes that all nonlinear terms, that is, terms involving products of field quantities or products of the derivatives of field quantities can be neglected.

Thus we can write the linearized versions of equations (1.1.10), (1.1.11), (1.1.12), (1.1.15) and (1.1.18) as:

$$\nu_\rho = 0 \quad , \quad \nu_z = 0 \quad , \tag{1.1.19}$$

$$\nabla^2 \lambda = 0 \quad , \tag{1.1.20}$$

$$\psi_\rho = \rho \Phi_z \quad , \quad \psi_z = -\rho \Phi_\rho , \tag{1.1.21}$$

$$\nabla^2 \Phi = 0 \quad . \tag{1.1.22}$$

The symmetries of physical systems often lead to the use of "preferred" coordinate systems, that is, coordinate systems in which the field equations take a particularly simple form. Since a disk is a degenerate oblate spheroid it is likely that oblate spheroidal coordinates will simplify the description of the disk.

Oblate spheroidal coordinates are defined by [5]

$$\rho = A(1+\varepsilon^2)^{\frac{1}{2}} (1-\eta^2)^{\frac{1}{2}} \qquad z = A\varepsilon\eta \tag{1.1.23}$$

where $0 \leq \varepsilon < \infty$, $-1 \leq \eta \leq 1$. These coordinates are ideally suited for a disk since the disk surface is given by $\varepsilon = 0$.

The Laplacian of an arbitrary function F has the form

$$\nabla^2 F = \frac{1}{A^2(\epsilon^2+\eta^2)} \left[\frac{\partial}{\partial\epsilon}(1+\epsilon^2)\frac{\partial F}{\partial\epsilon} + \frac{\partial}{\partial\eta}(1-\eta^2)\frac{\partial R}{\partial\eta}\right] . \tag{1.1.24}$$

Separation of variables leads to the following solution to Laplaces equation

$$F = \sum_{\eta=0}^{\infty} a_n Q_n(i\varepsilon) P_n(\eta) \tag{1.1.25}$$

where P_n and Q_n are respectively Legendre functions of the first and second kind. We shall later use expression (1.1.25) in the description of the linearized field of the disk.

II. The Linearized Solution.

The linearized equations written in terms of oblate spheroidal coordinates take the form

$$\nu_\eta = 0 \quad , \quad \nu_\varepsilon = 0 \quad , \tag{1.2.1}$$

$$\nabla^2\lambda = 0 \quad , \tag{1.2.2}$$

$$\psi_\varepsilon = A(1-\eta^2)\Phi_\eta \quad , \quad \psi_\eta = -A(1+\varepsilon^2)\Phi_\varepsilon \quad , \tag{1.2.3}$$

$$\nabla^2\Phi = 0 \quad . \tag{1.2.4}$$

To simplify the problem we shall make the following assumptions:

(i) the angular velocity ω as seen by an observer at infinity is constant.

(ii) the gravitational and centrifugal forces on each particle are equal.

Since (ii) implies that the pressure is zero in the disk, we can write the solution to Laplace's equation (1.2.2) as

$$\lambda(x,y,z) = - \int_S \frac{\sigma(x',y')}{R} \, dx'dy' \qquad (1.2.5)$$

where $\sigma(x',y')$ is the surface density of the disk and R is the distance from source point (x',y') to observation point (x,y,z) .

Using expression (1.2.5) for λ we can evaluate $[\frac{\partial\lambda}{\partial z}]_{z=0}$, the jump in the normal derivative of λ due to the surface layer σ . We obtain

$$[\frac{\partial\lambda}{\partial z}]_{\substack{z=0 \\ \rho\leq A}} = 4\pi\sigma \quad , \qquad (1.2.6)$$

$$[\frac{\partial\lambda}{\partial z}]_{\substack{z=0 \\ \rho\geq A}} = 0 \qquad (1.2.7)$$

which are equivalent to

$$(\frac{\partial\lambda}{\partial\varepsilon})_{\varepsilon=0} = 2\pi A\sigma\eta \, , \qquad (1.2.8)$$

$$\left(\frac{\partial\lambda}{\partial\eta}\right)_{\eta=0} = 0 \quad . \tag{1.2.9}$$

Conditions (i) and (ii) may be expressed as

$$\omega^2 = -\frac{1}{\rho}\left(\frac{\partial\lambda}{\partial\rho}\right) = \text{constant} \tag{1.1.10}$$

for $z = 0$, $\rho \leq A$.

In terms of oblate spheroidal coordinates (1.2.10) has the form

$$\omega^2 = -\frac{1}{A^2\eta}\left(\frac{\partial\lambda}{\partial\eta}\right)_{\varepsilon=0} = \text{constant} \quad . \tag{1.2.11}$$

As we observed earlier λ takes the form

$$\lambda = \sum_{n=0}^{\infty} a_n P_n(\eta) Q_n(i\varepsilon) \quad . \tag{1.2.12}$$

Condition (1.2.9) implies $a_n = 0$ for n odd, and (1.2.11) implies $a_n = 0$ for $n > 2$. The coefficients a_o and a_2 can be determined from (1.2.8) and (1.2.11) and the requirement σ be finite for $(\eta=0)$. The result is

$$\lambda = -\frac{4}{3}\pi\sigma_o A\left\{\frac{Q_o(i\varepsilon)}{(-i)} P_o(\eta) - \frac{Q_2(i\varepsilon)}{(-i)} P_2(\eta)\right\} , \tag{1.2.13}$$

$$\sigma = 2\sigma_o\eta = 2\sigma_o\left(1 - \frac{\rho^2}{A^2}\right)^{1/2} , \tag{1.2.14}$$

where

$$\sigma_o = \frac{A\omega^2}{\pi^2} \quad . \tag{1.2.15}$$

In the linear approximation the $g_{\alpha\beta}$ are given by [2]

$$g_{\alpha\beta} = \eta_{\alpha\beta} + 4 \int \frac{(S_{\alpha\beta} - \frac{1}{2}\eta_{\alpha\beta}S)}{R} \, dx^1 dx^2$$

where $\eta_{\alpha\beta}$ is the Minkowski metric and $S_{\alpha\beta}$ the energy momentum tensor of the matter distribution.

According to the physical interpretation of the energy momentum tensor we have

$$\begin{aligned} S_{tx} &= \sigma(x,y)\,\omega(x,y)\,y \quad , \\ S_{ty} &= -\sigma(x,y)\,\omega(x,y)\,x \quad , \\ S_{tz} &= 0 \quad . \end{aligned} \tag{1.2.16}$$

Hence

$$\begin{aligned} g_{tx} &= \int \frac{4\,\sigma(x',y')\,\omega(x',y')\,y'}{R} \, dx'dy' \quad , \\ g_{ty} &= \int - \frac{4\,\sigma(x',y')\,\omega(x',y')\,x'}{R} \, dx'dy' \quad , \\ g_{tz} &= 0 \quad . \end{aligned} \tag{1.2.17}$$

From the above expressions we can determine the jump in the normal derivative of g_{tx} , g_{ty} and g_{tz} . One concludes:

$$
\begin{aligned}
\left[\frac{\partial g_{tx}}{\partial z}\right]_{z=0} &= - 16 \pi y \bar{\sigma} \omega \quad , \\
\left[\frac{\partial g_{ty}}{\partial z}\right]_{z=0} &= 16 \pi x \sigma \omega \quad , \qquad (1.2.18) \\
\left[\frac{\partial g_{tz}}{\partial z}\right]_{z=0} &= 0 \quad .
\end{aligned}
$$

However

$$\left[\frac{\partial g_{t\phi}}{\partial z}\right]_{z=0} d\phi = \left[\frac{\partial g_{tb}}{\partial z}\right]_{z=0} dx^b \qquad (1.2.19)$$

and in the linear approximation $g_{t\phi} = \psi$, so (1.2.18) leads to the result

$$\left(\frac{\partial \psi}{\partial z}\right)_{z=0+} = 32 \pi \sigma \omega \rho^2 \quad . \qquad (1.2.20)$$

From (1.2.3) one can now obtain the appropriate boundary condition for Φ , namely:

$$\left(\frac{\partial \Phi}{\partial \eta}\right)_{\varepsilon=0} = 16 \pi \sigma_o \omega A^2 \eta^2 \quad , \qquad (1.2.21)$$

$$\left(\frac{\partial \Phi}{\partial \varepsilon}\right)_{\eta=0} = 0 \quad . \qquad (1.2.22)$$

If we assume that Φ has the form (1.1.25) then the above conditions imply

$$\Phi = \frac{16}{5} \pi \sigma_o A^2 \omega \{-P_1(\eta) Q_1(i\varepsilon) + P_3(\eta) Q_3(i\varepsilon)\} \quad . \tag{1.2.23}$$

At this point it seems convenient to introduce the dimensionless parameter

$$\alpha = \frac{6\omega A}{5} \tag{1.2.24}$$

also, for economy we define

$$\begin{aligned} \kappa &= \frac{50}{27\pi} \quad , \qquad Q_{2n+1}(i\varepsilon) = \tilde{Q}_{2n+1}(\varepsilon) \quad , \\ \frac{Q_{2n}(i\varepsilon)}{(-i)} &= \tilde{Q}_{2n}(\varepsilon) \quad , \\ A_o &= \kappa(P_o\tilde{Q}_o - P_2\tilde{Q}_2) \quad , \\ B_o &= -\kappa(-P_1\tilde{Q}_1 + P_3\tilde{Q}_3) \quad . \end{aligned} \tag{1.2.25}$$

In this notation the Ernst function (1.1.16) can be written in the form

$$\mathcal{E} = 1 - \alpha^2 A_o + i\, \alpha^3 B_o \tag{1.2.26}$$

which can be regarded as the first few terms of a power series expansion.

It turns out that this point of view is useful in determining higher order approximations.

To complete the specification of the linearized field we must determine ψ and ν. According to (1.2.3) we have

$$\frac{\partial\psi}{\partial\xi} = A(1-\eta^2)\,\frac{\partial\Phi}{\partial\eta}$$
$$\frac{\partial\psi}{\partial\eta} = -A(1+\varepsilon^2)\,\frac{\partial\Phi}{\partial\varepsilon} \tag{1.2.27}$$

which determines ψ up to an arbitrary constant. By the assumption of asymptotic flatness (i.e. $\varepsilon \to \infty \Rightarrow \psi \to 0$) we obtain ψ in the form

$$\psi = -\frac{\alpha^3 \kappa A}{2}\,(1+\varepsilon^2)(1-\eta^2)\,\{\tilde{Q}'_1(\varepsilon)\,P'_1(\eta) - \frac{1}{6}\,\tilde{Q}'_3(\varepsilon)\,P'_3(\eta)\} \tag{1.2.28}$$

where prime denotes differentiation with respect to the argument of the function.

The solution to the ν-equation (1.2.1) that is consistent with asymptotic flatness is

$$\nu = 0 \quad . \tag{1.2.29}$$

The metric (1.1.9) in terms of oblate spherical coordinates has the form

$$ds^2 = e^{2(\nu-\lambda)}\,[A^2(\varepsilon^2+\eta^2)\{\,\frac{d\varepsilon^2}{1+\varepsilon^2} + \frac{d\eta^2}{1-\eta^2}\,\}]$$
$$+\,e^{-2\lambda}\,A^2(1+\varepsilon^2)(1-\eta^2)d\phi^2 - e^{2\lambda}\,(d\tau-\psi d\phi)^2 \quad . \tag{1.2.30}$$

The linearized version of the above metric is

$$ds^2 = (1-2\lambda)\ [A^2(\varepsilon^2+\eta^2)\,\{\frac{d\varepsilon^2}{1+\varepsilon^2} + \frac{d\eta^2}{1-\eta^2}\}]\ + \tag{1.2.31}$$

$$+\ (1-2\lambda)\ A^2(1+\varepsilon^2)(1-\eta^2)\ d\phi^2 - (1+2\lambda)\ d\tau^2 + 2\psi d\phi d\tau\ .$$

More explicitly we conclude:

$$g_{\varepsilon\varepsilon} = \frac{A^2(\varepsilon^2+\eta^2)}{1+\varepsilon^2}\ \{1 + \kappa\ \alpha^2\ [\tilde{Q}_o\ P_o - \tilde{Q}_2\ P_2]\}$$

$$g_{\eta\eta} = \frac{A^2(\varepsilon^2+\eta^2)}{1-\eta^2}\ \{1 + \kappa\ \alpha^2\ [\tilde{Q}_o\ P_o - \tilde{Q}_2\ P_2]\}$$

$$g_{\phi\phi} = A^2(1+\varepsilon^2)(1-\eta^2)\ \{1 + \kappa\ \alpha^2\ [\tilde{Q}_o\ P_o - \tilde{Q}_2\ P_2]\} \tag{1.2.32}$$

$$g_{\tau\tau} = -\{1 - \kappa\ \alpha^2\ [\tilde{Q}_o\ P_o - \tilde{Q}_2\ P_2]\}$$

$$g_{\tau\phi} = g_{\phi\tau} = -\ \frac{\alpha^3\kappa A}{2}\ [(1+\varepsilon^2)(1-\eta^2)(\tilde{Q}_1' P_1' - \frac{1}{6}\ \tilde{Q}_3' P_2')]$$

where the terms inside the square brackets represent the deviation from flatness of the geometry.

CHAPTER II

The Surface Energy Tensor

I. Hypersurfaces in General Relativity [6].

Since we are concerned with a disk of infinitesimal thickness in a four-dimensional space-time we shall give a short discussion of the geometry of hypersurfaces.

In order to completely describe a hypersurface it is necessary to specify the intrinsic properties of the hypersurface (i.e., those properties which are independent of the imbedding space), and the so-called extrinsic properties which describe how the hypersurface is imbedded in the surrounding space.

The extrinsic geometry of a hypersurface can be described by a 3×3 symmetric tensor which is defined as follows.

Suppose we are given a four-dimensional Riemannian space-time V, in which is imbedded a smooth hypersurface $\sum$. Further suppose that $\sum$ is defined by

$$x^{\alpha} = f^{\alpha} (\varepsilon^{i}) \quad . \qquad (2.1.1)$$

The ε^{i} can be considered as intrinsic coordinates on the hypersurface.

An infinitesimal displacement in Σ

$$dx^{\alpha} = \frac{\partial f^{\alpha}}{\partial \varepsilon^{i}} \, d\varepsilon^{i} \tag{2.1.2}$$

defines the quantity $f^{\alpha}{}_{,i}$ which transforms as a contravariant four-vector for fixed i, and as a covariant surface-vector when α is fixed. Also the triad of four-vectors $(f^{\alpha}{}_{,1}, f^{\alpha}{}_{,2}, f^{\alpha}{}_{,3})$ are linearly independent and tangent to Σ.

If η^{α} is a space-like unit vector normal to Σ then we note that

$$\eta_{\alpha} \frac{\delta \eta^{\alpha}}{\delta \varepsilon_{i}} = 0 \tag{2.1.3}$$

that is the quantity

$$\frac{\delta \eta^{\alpha}}{\delta \varepsilon_{i}} \tag{2.1.4}$$

is (for each fixed i) a four-vector tangent to the surface. Hence there exists quantities $K_{i}{}^{j}$ such that

$$\frac{\delta \eta^{\alpha}}{\delta \varepsilon_{i}} = K_{i}{}^{j} f^{\alpha}{}_{,j} \quad . \tag{2.1.5}$$

K_{ij} is called the extrinsic curvature tensor of the hypersurface.

Since the surface metric tensor is given by

$$a_{ij} = f^{\alpha}{}_{,i}\, f^{\beta}{}_{,j}\, g_{\alpha\beta} \tag{2.1.6}$$

it follows from (2.1.5), (2.1.4) and (2.1.6) that

$$K_{ij} = -\eta_{\alpha} \frac{\delta(f^{\alpha}{}_{,i})}{\delta\, \varepsilon_{j}} \quad . \tag{2.1.7}$$

The extrinsic curvature tensor can be used to define a singular hypersurface or surface layer in general relativity as follows.

Consider a Riemannian space-time V which is constructed from two Riemannian space-times V^- , V^+ whose mutual boundary in V is given by a hypersurface $\sum$. Let η^{α} be the unit normal to $\sum$ and K_{ij}^{-} , K_{ij}^{+} the extrinsic curvatures of $\sum$ associated with its imbeddings in V^- , V^+ respectively.

The hypersurface $\sum$ is called a singular hypersurface or surface layer iff

$$\gamma_{ij} = K_{ij}^{+} - K_{ij}^{-} \neq 0 \quad . \tag{2.1.8}$$

The γ_{ij} defined above can now be used to define the surface energy tensor S_{ij} ,

$$- 8\pi\, S_{ij} = \gamma_{ij} - g_{ij}\, \gamma \, , \tag{2.1.9}$$

where $\gamma = g^{ab}\, \gamma_{ab}$.

For a justification of the name surface energy tensor we refer the reader to Israel [6].

The tensor S_{ij} defines three linearly independent eigenvectors one of which is timelike. Hence the surface density σ can be defined by the eigenvalue equation

$$S_a{}^b u^a = -\sigma u^b \quad , \tag{2.1.10}$$

$$u^a u_a = -1 \quad .$$

II. <u>The Surface Energy Tensor of the Disk.</u>

We now apply the previous results to the special case in which $\sum$ is the surface of a disk. In oblate spheroidal coordinates the surface is defined by $\varepsilon = 0$. We will use the following convention for the coordinates

$$\begin{aligned} x^\alpha &= (x^1,x^2,x^3,x^4) = (\varepsilon,\eta,\phi,\tau) \quad , \\ \varepsilon^i &= (\varepsilon^2,\varepsilon^3,\varepsilon^4) = (\eta,\phi,\tau) \quad . \end{aligned} \tag{2.2.1}$$

In terms of the above coordinates equation (2.1.7) reduces to

$$K_{ij} = -\eta_\alpha \Gamma^\alpha_{\lambda\mu} \delta^\lambda_i \delta^\mu_j . \tag{2.2.2}$$

The unit vector normal to the surface $\varepsilon = 0$ is given by

$$\eta_\alpha = \pm\ (A(1+\varepsilon^2)^{\frac{1}{2}}\ \eta\ e^{(\nu-\lambda)}\ ,\ -\ A\ \varepsilon\ (1-\eta^2)^{\frac{1}{2}}\ e^{(\nu-\lambda)},0,0)\quad ,$$

$$\eta^\alpha = \pm \left(\frac{(1+\varepsilon^2)^{\frac{1}{2}}\ \eta\ e^{-(\nu-\lambda)}}{A(\varepsilon^2+\eta^2)}\ ,\ -\ \frac{\varepsilon(1-\eta^2)^{\frac{1}{2}}\ e^{-(\nu-\lambda)}}{A(\varepsilon^2+\eta^2)}\ ,\ 0\ ,\ 0 \right) .$$

(2.2.3)

It follows that (2.2.2) reduces to

$$K_{ij} = -\eta_1\ \Gamma^1_{ij} - \eta_2\ \Gamma^2_{ij} \quad . \tag{2.2.4}$$

The contravariant metric is given by

$$g^{ij} = \begin{pmatrix} \frac{(1+\varepsilon^2)e^{-2(\nu-\lambda)}}{A^2(\varepsilon^2+\eta^2)} & 0 & 0 & 0 \\ 0 & \frac{(1-\eta^2)e^{-2(\nu-\lambda)}}{A^2(\varepsilon^2+\eta^2)} & 0 & 0 \\ 0 & 0 & \frac{e^{2\lambda}}{A^2(1+\varepsilon^2)(1-\eta^2)} & \frac{\psi\ e^{2\lambda}}{A^2(1+\varepsilon^2)(1-\eta^2)} \\ 0 & 0 & \frac{\psi\ e^{2\lambda}}{A^2(1+\varepsilon^2)(1-\eta^2)} & \frac{e^{2\lambda}\ \psi^2}{A^2(1+\varepsilon^2)(1-\eta^2)} - e^{-2\lambda} \end{pmatrix}$$

From (1.2.32), (2.2.3) and (2.2.5) we can calculate the exact extrinsic curvatures for the disk using (2.2.2) which we list below:

$$K_\eta{}^\eta = \frac{(\nu-\lambda)_\varepsilon \, e^{-(\nu-\lambda)}}{A\eta}$$

$$K_\phi{}^\phi = \frac{e^{-(\nu-\lambda)}}{A^2\eta^2} \left\{ -\lambda_\varepsilon A\eta - \frac{e^{4\lambda}\psi\psi_\varepsilon \eta}{2A(1-\eta^2)} \right\} ,$$

$$K_\tau{}^\tau = -K_\phi{}^\phi \tag{2.2.6}$$

$$K_\phi{}^\tau = \frac{e^{-(\nu-\lambda)}}{A^2\eta^2} \left\{ -2A\lambda_\varepsilon \psi\eta - \frac{e^{4\lambda}\eta\psi^2\psi_\varepsilon}{2A(1-\eta^2)} - \frac{A\eta\psi_\varepsilon}{2} \right\} ,$$

$$K_\tau{}^\varepsilon = \frac{e^{-(\nu-\lambda)}}{A^2\eta^2} \left\{ \frac{e^{4\lambda}\eta\,\psi_\varepsilon}{2A(1-\eta^2)} \right\} ,$$

$$K_\eta{}^\phi = K_\phi{}^\eta = K_\eta{}^\tau = K_\tau{}^\eta = 0 \quad .$$

(Subscripts on functions indicate partial differentiation with respect to that subscript.)

Now that general expressions for the extrinsic curvatures have been obtained, equation (2.1.9) can be used to determine the surface energy tensor S_{ij} . In the case of a disk $K_a{}^{b-} = -K_a{}^{b+}$. Thus

$$S_i{}^j = -\frac{1}{4\pi}(K_i{}^j - \delta_i{}^j K) \quad . \tag{2.2.7}$$

From (2.2.6) we obtain the following exact expressions for $S_i{}^j$:

$$S_\phi{}^\phi = \frac{e^{-(\nu-\lambda)}}{4\pi\ A^2\eta^2}\left\{A\eta\ \nu_\varepsilon + \frac{\eta\ e^{4\lambda}\ \psi\psi_\varepsilon}{2A(1-\eta^2)}\right\} ,$$

$$S_\tau{}^\tau = \frac{e^{-(\nu-\lambda)}}{4\pi\ A^2\eta^2}\left\{-2A\eta\ \lambda_\varepsilon + A\eta\ \nu_\varepsilon - \frac{\eta\ e^{4\lambda}\psi\psi_\varepsilon}{2A(1-\eta^2)}\right\} , \qquad (2.2.8)$$

$$S_\phi{}^\tau = \frac{e^{-(\nu-\lambda)}}{4\pi\ A^2\eta^2}\left\{2A\eta\ \psi\lambda_\varepsilon + \frac{\eta\ e^{4\lambda}\ \psi^2\psi_\varepsilon}{2A(1-\eta^2)} + \frac{A\eta\psi_\varepsilon}{2}\right\} ,$$

$$S_\tau{}^\phi = -\frac{e^{-(\nu-\lambda)}}{4\pi\ A^2\eta^2}\left\{\frac{\eta\ e^{4\lambda}\ \psi_\varepsilon}{2A(1-\eta^2)}\right\} ,$$

$$S_\eta{}^\eta = S_\eta{}^\phi = S_\phi{}^\eta = S_\eta{}^\tau = S_\tau{}^\eta = 0 .$$

The linearized expressions for the surface energy tensor are:

$$S_\phi{}^\phi = \frac{\nu_\varepsilon}{4\pi A\eta} ,$$

$$S_\tau{}^\tau = -\frac{\lambda_\varepsilon}{2\pi A\eta} + \frac{\nu_\varepsilon}{4\pi A\eta} ,$$

$$S_\phi{}^\tau = \frac{\psi_\varepsilon}{8\pi A\eta} , \qquad (2.2.9)$$

$$S_\tau{}^\phi = -\frac{\psi_\varepsilon}{8\pi A^3\eta(1-\eta^2)} ,$$

$$S_\eta{}^\eta = S_\nu{}^\phi = S_\phi{}^\eta = S_\eta{}^\tau = S_\tau{}^\eta = 0 .$$

Using the linearized expressions for λ , ψ and ν given in (B.1) it follows that the explicit values of $S_i{}^j$ for the disk are:

$$S_\tau{}^\tau = -\frac{\alpha^2 3\kappa\eta}{4\pi A} \quad ,$$

$$S_\tau{}^\phi = -\frac{5\kappa\alpha^3\eta}{8\pi A^2} \quad , \tag{2.2.10}$$

$$S_\phi{}^\tau = \frac{5\kappa\alpha^3\eta(1-\eta^2)}{8\pi} \quad ,$$

$$S_\phi{}^\phi = S_\eta{}^\eta = S_\eta{}^\phi = S_\phi{}^\eta = S_\eta{}^\tau = S_\tau{}^\eta = 0 \quad .$$

The above explicit expressions for the surface energy tensor for the linearized gravitational field can be used to determine the surface rest mass density. According to equation (2.1.10) we must solve

$$\det\,(S_a{}^b + \sigma\,\delta_a{}^b) = 0 \quad . \tag{2.2.11}$$

The exact solution is

$$\sigma = -\frac{1}{2}\,S_\tau{}^\tau - \frac{1}{2}\,(S_\tau{}^{\tau^2} + 4S_\tau{}^\phi\;S_\phi{}^\tau)^{\frac{1}{2}} \quad . \tag{2.2.12}$$

This reduces in the linear approximation i.e. $o(\alpha^3)$ to

$$\sigma = -\,S_\tau{}^\tau = 2\,\sigma_o\,\eta \tag{2.2.13}$$

which is consistent with (1.2.14).

We can also obtain the angular velocity of an observer at infinity $\omega = \frac{d\phi}{d\tau}$ by observing that $u^\eta = u^\varepsilon = 0$ since there is no flow

of matter in these directions and

$$u^{\phi} = \frac{d\phi}{ds} = \omega\ u^{\tau} \quad . \tag{2.2.14}$$

Thus from (2.1.10) we obtain

$$\omega = \frac{S_{\tau}^{\ \phi}}{\omega S_{\phi}^{\ \tau} + S_{\tau}^{\ \tau} - S_{\phi}^{\ \phi}} \tag{2.2.15}$$

or

$$\omega = - \frac{S_{\tau}^{\ \tau} + ({S_{\tau}^{\ \tau}}^{2} + 4S_{\phi}^{\ \tau}\ S_{\tau}^{\ \phi})^{\frac{1}{2}}}{2\ S_{\phi}^{\ \tau}} \quad . \tag{2.2.16}$$

Using (2.2.10) we obtain good to order α^3

$$\omega = \frac{5\alpha}{6A} \tag{2.2.17}$$

which is in agreement with (1.2.24).

CHAPTER III

Higher Order Approximations

I. Outline of Method.

In chapter I we noted that the Ernst function appeared to have a natural expansion in terms of the parameter α . By choosing a particular form of the Ernst function it turns out that with very little effort the second order approximation can be obtained.

Recall that in chapter I, equation (1.1.16), we introduced the Ernst function

$$\mathcal{E} = e^{2\lambda} + i\,\Phi \quad .$$

It is advantageous to rewrite the above expression in terms of two real valued functions $A(\varepsilon,\eta)$, $B(\varepsilon,\eta)$ as follows

$$\mathcal{E} = e^{-\alpha^2(A+i\alpha B)} = e^{2\lambda} + i\,\Phi \ . \qquad (3.1.1)$$

With the above form of the Ernst function the field equation (1.1.17) reduces to two nonlinear coupled equations:

$$\nabla^2 A = -\,2\alpha^3 \tan\,(\alpha^3 B)[(\nabla A)\cdot(\nabla B)] \quad , \qquad (3.1.2)$$

$$\nabla^2 B = \alpha \tan\,(\alpha^3 B)[(\nabla A)^2 - \alpha^2(\nabla B)^2] \ . \qquad (3.1.3)$$

These equations are ideal for approximation purposes since A and B are harmonic to orders α^5 and α^3 respectively.

The relationship between the functions A , B and Φ , λ is from (3.1.1)

$$e^{2\lambda} = e^{-\alpha^2 A} \cos (\alpha^3 B) \quad , \tag{3.1.4}$$

$$\Phi = -e^{-\alpha^2 A} \sin (\alpha^3 B) \quad . \tag{3.1.5}$$

If we now assume A and B have expansions of the form

$$\begin{aligned} A &= A_o + \alpha^{\ell} A_1 + \alpha^{\mu} A_2 + \cdots \quad , \\ B &= B_o + \alpha^{\beta} B_1 + \alpha^{\gamma} B_2 + \cdots \end{aligned} \tag{3.1.6}$$

then from equations (3.1.4) and (3.1.5) we see that $-\frac{A_o}{2}$ and $-B_o$ are respectively the λ and Φ of the linearized solution, thus A_o and B_o are harmonic.

The unknown powers of the expansion parameter α in equation (3.1.6) can be determined from (3.1.2) and (3.1.3) the result is:

$$\begin{aligned} A &= A_o + \alpha^6 A_1 + \alpha^{10} A_2 + \cdots \quad , \\ B &= B_o + \alpha^4 B_1 + \alpha^6 B_2 + \cdots \end{aligned} \tag{3.1.7}$$

where A_1 , B_1 , B_2 satisfy the following equations.

$$\nabla^2 A_1 = - 2 B_o (\nabla A_o)(\nabla B_o) \quad , \tag{3.1.9}$$

$$\nabla^2 B_1 = B_o (\nabla A_o)^2 \quad , \tag{3.1.10}$$

$$\nabla^2 B_2 = - B_o (\nabla B_o)^2 \quad . \tag{3.1.11}$$

The relationship between A , B and λ , Φ of the two different representations of the Ernst function can now be determined from equation (3.1.4), (3.1.5), (3.1.7) and (3.1.8), we obtain:

$$\lambda = \alpha^2 \lambda_o + \alpha^6 \lambda_1 + \alpha^8 \lambda_2 + \cdots , \tag{3.1.12}$$

$$\Phi = \alpha^3 \Phi_o + \alpha^5 \Phi_1 + \alpha^7 \Phi_2 + \cdots \tag{3.1.13}$$

where

$$\begin{aligned} \lambda_o &= - \frac{A_o}{2} \quad , \quad & \Phi_o &= - B_o \quad , \\ \lambda_1 &= - \frac{B_o^2}{4} \quad , \quad & \Phi_1 &= A_o B_o \quad , \\ \lambda_2 &= - \frac{A_1}{2} \quad , \quad & \Phi_2 &= - B_1 - \frac{A_o^2}{2} B_o \quad . \end{aligned} \tag{3.1.14}$$

It is evident from the above expressions that A_o and B_o determine the gravitational field to order α^6 .

Recall that ψ is determined by the equation (1.1.15) which in oblate spheroidal coordinates take the form

$$\frac{\partial\psi}{\partial\varepsilon} = e^{-4\lambda} A(1-\eta^2) \frac{\partial\Phi}{\partial\eta} \quad , \tag{3.1.15}$$

$$\frac{\partial\psi}{\partial\eta} = -e^{-4\lambda} A(1+\epsilon^2) \frac{\partial\Phi}{\partial\epsilon} \quad . \tag{3.1.16}$$

Hence from equations (3.1.12) and (3.1.13) we conclude that ψ has a similiar expansion

$$\psi = \alpha^3\psi_o + \alpha^5\psi_1 + \alpha^7\psi_2 + \cdots \tag{3.1.17}$$

where ψ_o , ψ_1 , and ψ_2 are to be determined from the following equations:

$$\begin{aligned} \frac{\partial\psi_o}{\partial\varepsilon} &= A(1-\eta^2) \frac{\partial\Phi_o}{\partial\eta} \quad , \\ \frac{\partial\psi_o}{\partial\eta} &= - A(1+\varepsilon^2) \frac{\partial\Phi_o}{\partial\varepsilon} \quad , \end{aligned} \tag{3.1.18}$$

$$\begin{aligned} \frac{\partial\psi_1}{\partial\eta} &= - A(1+\varepsilon^2) \left[\frac{\partial\Phi_1}{\partial\varepsilon} + 2 A_o \frac{\partial\Phi_o}{\partial\varepsilon}\right] \quad , \\ \frac{\partial\phi_1}{\partial\varepsilon} &= A(1-\eta^2) \left[\frac{\partial\Phi_1}{\partial\eta} + 2 A_o \frac{\partial\Phi_o}{\partial\eta}\right] \quad , \end{aligned} \tag{3.1.19}$$

$$\begin{aligned} \frac{\partial\psi_2}{\partial\varepsilon} &= A(1-\eta^2) \left[\frac{\partial\Phi_2}{\partial\eta} + 2 A_o \frac{\partial\Phi_1}{\partial\eta} + 2 A_o^2 \frac{\partial\Phi_o}{\partial\eta}\right] \quad , \\ \frac{\partial\psi_2}{\partial\eta} &= - A(1+\varepsilon^2) \left[\frac{\partial\Phi_2}{\partial\varepsilon} + 2 A_o \frac{\partial\Phi_1}{\partial\varepsilon} + 2 A_o^2 \frac{\partial\Phi_o}{\partial\varepsilon}\right] \quad . \end{aligned} \tag{3.1.20}$$

Using the expressions for Φ_o , Φ_1 , Φ_2 from (3.1.14), equations (3.1.19) and (3.1.20) can be rewritten

$$\frac{\partial\psi_1}{\partial\varepsilon} = A(1-\eta^2)\,[\frac{\partial A_o}{\partial\eta} B_o - \frac{\partial B_o}{\partial\eta} A_o] \quad ,$$

$$\frac{\partial\psi_1}{\partial\eta} = -A(1+\varepsilon^2)\,[\frac{\partial A_o}{\partial\varepsilon} B_o - \frac{\partial B_o}{\partial\varepsilon} A_o] \quad , \tag{3.1.21}$$

$$\frac{\partial\psi_2}{\partial\varepsilon} = -A(1-\eta^2)\,[\frac{1}{2}\frac{\partial B_o}{\partial\eta} A_o{}^2 - A_o B_o \frac{\partial A_o}{\partial\eta} + \frac{\partial B_1}{\partial\eta}] \quad ,$$

$$\frac{\partial\psi_2}{\partial\eta} = A\,(1+\varepsilon^2)\,[\frac{1}{2}\frac{\partial B_o}{\partial\varepsilon} A_o{}^2 - A_o B_o \frac{\partial A_o}{\partial\varepsilon} + \frac{\partial B_1}{\partial\varepsilon}] \quad . \tag{3.1.22}$$

Finally in order to complete the description of the gravitational field it is necessary to calculate ν .

Rewriting the ν-equations, (1.1.10) and (1.1.11) in terms of oblate spheroidal coordinates we obtain

$$\nu_\varepsilon = \frac{\varepsilon(1+\varepsilon^2)(1-\eta^2)}{(\varepsilon^2+\eta^2)}\,(\frac{\partial\lambda}{\partial\varepsilon})^2 - \frac{\varepsilon(1-\eta^2)^2}{(\varepsilon^2+\eta^2)}\,(\frac{\partial\lambda}{\partial\eta})^2 - \frac{2(1+\varepsilon^2)(1-\eta^2)\eta}{(\varepsilon^2+\eta^2)}\,(\frac{\partial\lambda}{\partial\varepsilon})(\frac{\partial\lambda}{\partial\eta}) \; +$$

$$+\, e^{4\lambda}[\frac{\varepsilon}{4A^2(\varepsilon^2+\eta^2)}\,(\frac{\partial\psi}{\partial\varepsilon})^2 + \frac{\varepsilon(1-\eta^2)}{4\,A^2(\varepsilon^2+\eta^2)(1+\varepsilon^2)}\,(\frac{\partial\psi}{\partial\eta})^2 + \frac{\eta}{2A^2(\varepsilon^2+\eta^2)}\,(\frac{\partial\psi}{\partial\eta})(\frac{\partial\psi}{\partial\varepsilon})] \; , \tag{3.1.23}$$

$$\nu_\eta = \frac{(1+\varepsilon^2)^2\eta}{(\varepsilon^2+\eta^2)}\,(\frac{\partial\lambda}{\partial\varepsilon})^2 - \frac{(1+\varepsilon^2)(1-\eta^2)\eta}{(\varepsilon^2+\eta^2)}\,(\frac{\partial\lambda}{\partial\eta})^2 + \frac{2\varepsilon(1+\varepsilon^2)(1-\eta^2)}{(\varepsilon^2+\eta^2)}\,(\frac{\partial\lambda}{\partial\varepsilon})(\frac{\partial\lambda}{\partial\eta}) \; +$$

$$+ e^{4\lambda} \left[- \frac{\eta(1+\varepsilon^2)}{4A^2(1-\eta^2)(\varepsilon^2+\eta^2)} \left(\frac{\partial\psi}{\partial\varepsilon}\right)^2 + \frac{\eta}{4A^2(\varepsilon^2+\eta^2)} \left(\frac{\partial\psi}{\partial\eta}\right)^2 - \frac{\varepsilon}{2A^2(\varepsilon^2+\eta^2)} \left(\frac{\partial\psi}{\partial\varepsilon}\right)\left(\frac{\partial\psi}{\partial\eta}\right) \right] . \tag{3.1.24}$$

II. The Gravitational Field to Order α^6 .

As we noted earlier knowledge of the linearized gravitational field, i.e. A_o , B_o is sufficient to determine the gravitational field to order α^6 .

Equations (3.1.12), (3.1.14) and (1.2.25) give

$$\lambda_1 = - \frac{B_o^{\ 2}}{4} ,$$

$$\lambda_1 = - \frac{\kappa^2}{4} \{P_3^{\ 2}\tilde{Q}_3^{\ 2} + P_1^{\ 2}\tilde{Q}_1^{\ 2} - 2P_3P_1\tilde{Q}_3\tilde{Q}_1\} . \tag{3.2.1}$$

To obtain ψ_1 it is necessary to integrate equations (3.1.21). Using the expressions for A_o and B_o given by (1.2.13) and (1.2.25) respectively leads to the result

$$\psi_1 = - \kappa^2 A (1+\varepsilon^2)(1-\eta^2) \{ \frac{1}{12} P_3{}' (\tilde{Q}_3\tilde{Q}_o{}' - \tilde{Q}_o\tilde{Q}_3{}')$$

$$+ \frac{1}{2} (\tilde{Q}_1{}'\tilde{Q}_o - \tilde{Q}_1\tilde{Q}_o{}') + \frac{1}{6} (P_3P_2{}' - P_3{}'P_2)(\tilde{Q}_3\tilde{Q}_2{}' - \tilde{Q}_3{}'\tilde{Q}_2)$$

$$+ \frac{1}{4} (P_1{}'P_2 - P_2{}'P_1)(\tilde{Q}_2\tilde{Q}_1{}' - \tilde{Q}_2{}'\tilde{Q}_1)\} \tag{3.2.2}$$

where the constant of integration is zero by the requirement of asymptotic flatness.

The functions ν_1 and ν_2 which correspond to $\lambda = \alpha^2\lambda_o$ and $\psi = \alpha^3\psi_o$ are obtained by the integration of (1.1.23) and (1.1.24). The result is given in appendix A.

To summarize the preceding results we have

$$\begin{aligned} \lambda &= \alpha^2\lambda_o + \alpha^6\lambda_1 , \\ \psi &= \alpha^3\psi_o + \alpha^5\psi_1 , \\ \nu &= \alpha^4\nu_1 + \alpha^6\nu_2 \end{aligned} \qquad (3.2.3)$$

where λ_o , ψ_o , λ_1 , ψ_1 , ν_1 and ν_2 are given in equations (1.2.13), (1.2.28), (3.2.1), (3.2.2), (A,1) and (A,2) respectively.

In terms of the above defined functions the gravitational potentials correct to order α^6 take the form

$$g_{\varepsilon\varepsilon} = \frac{A^2(\varepsilon^2+\eta^2)}{(1+\varepsilon^2)} \{1 - 2\alpha^2\lambda_o + 2\alpha^4(\nu_1+\lambda_o{}^2) + \alpha^6(2\nu_2 - 2\lambda_1 - \tfrac{4}{3}\lambda_o{}^3)\} ,$$

$$g_{\eta\eta} = \frac{A^2(\varepsilon^2+\eta^2)}{(1-\eta^2)} \{1 - 2\alpha^2\lambda_o + 2\alpha^4(\nu_1+\lambda_o{}^2) + \alpha^6(2\nu_2 - 2\lambda_1 - \tfrac{4}{3}\lambda_o{}^3)\} ,$$

$$g_{\phi\phi} = A^2(1+\varepsilon^2)(1-\eta^2) \{1 - 2\alpha^2\lambda_o + 2\alpha^4\lambda_o{}^2 + \alpha^6(-2\lambda_1 - \tfrac{4}{3}\lambda_o{}^3)\} - \alpha^6\psi_o{}^2 ,$$

$$g_{\tau\tau} = - \{1 + 2\alpha^2\lambda_o + 2\alpha^4\lambda_o^{\ 2} + \alpha^6 \ (2\lambda_1 + \frac{4}{3}\lambda_o^{\ 3})\} \quad ,$$

$$g_{\tau\phi} = g_{\phi\tau} = \alpha^3\psi_o + \alpha^5 \ (2\lambda_o\psi_o + \psi_1) \quad . \qquad (3.2.4)$$

III. The Surface Energy Tensor to Order α^6 .

The functions λ , ψ and ν can now be used to calculate the surface energy tensor by the technique used in chapter II.

Exact expressions for the surface energy tensor of the disk are given in equation (2.2.8). Neglecting all terms of higher order than α^6 one arrives at the following expressions:

$$S_\phi^{\ \phi} = \frac{1}{4\pi A^2\eta^2} \left\{\alpha^4 \ A\eta \ \nu_{1\varepsilon} + \alpha^6(A\eta \ \nu_{2\varepsilon} + A\eta \ \nu_{1\varepsilon}\lambda_o + \frac{\eta \ \psi_o\psi_{o\varepsilon}}{2A(1-\eta^2)})\right\} \quad ,$$

$$S_\tau^{\ \tau} = \frac{1}{4\pi A^2\eta^2} \ \{ - 2A\eta \ \alpha^2\lambda_{o\varepsilon} + \alpha^4(-2A\eta \ \lambda_{o\varepsilon}\lambda_o + A\eta \ \nu_{1\varepsilon}) \ +$$

$$+ \alpha^6(-2A\eta \ \lambda_{1\varepsilon} - \frac{\eta \ \psi_o\psi_{o\varepsilon}}{2A(1-\eta^2)} + 2A\eta \ \lambda_{o\varepsilon}\nu_1 - A\eta \ \lambda_{o\varepsilon}\lambda_o^{\ 2} \ +$$

$$+ A\eta \ \nu_{2\varepsilon} + A\eta \ \nu_{1\varepsilon}\lambda_o\} \quad , \qquad (3.3.1)$$

$$S_\phi^{\ \tau} = + \frac{1}{4\pi A^2\eta^2} \left\{\alpha^3(\frac{A\eta \ \psi_{o\varepsilon}}{2}) + \alpha^5(2A\eta \ \lambda_{o\varepsilon}\psi_o + \frac{A\eta \ \psi_{1\varepsilon}}{2} + \frac{A\eta \ \psi_{o\varepsilon}\lambda_o}{2})\right\} \ ,$$

$$S_{\tau}{}^{\phi} = -\frac{1}{4\pi A^2\eta^2}\left\{\frac{\alpha^3\eta\,\psi_{o\varepsilon}}{2A(1-\eta^2)} + \alpha^5\left(\frac{\eta\psi_{1\varepsilon}}{2A(1-\eta^2)} + \frac{5\eta\lambda_o\psi_{o\varepsilon}}{2A(1-\eta^2)}\right)\right\} ,$$

$$S_{\eta}{}^{\eta} = S_{\eta}{}^{\phi} = S_{\phi}{}^{\eta} = S_{\tau}{}^{\eta} = S_{\eta}{}^{\tau} = 0 .$$

The fact that $S_{\eta}{}^{\eta} = 0$ is identically zero (i.e. no pressure in the disk) follows from the assumption that $R_{\phi}{}^{\phi} + R_{\tau}{}^{\tau} = 0$ which was used in deriving the metric (1.1.9) (see Synge [2], P. 317).

Using the explicit expressions for λ_o , ψ_o , λ_1 , ψ_1 , ν_1 and ν_2 listed in appendix B, we obtain the following components of the surface energy tensor:

$$S_{\phi}{}^{\phi} = \frac{\alpha^4 9\kappa^2\eta(1-\eta^2)}{32A} + \frac{25\alpha^6\kappa^2\eta(1-\eta^2)(4\eta^2-3)}{128A} ,$$

$$S_{\tau}{}^{\tau} = -\frac{\alpha^2 3\kappa\eta}{4\pi A} + \frac{\alpha^4 9\kappa^2\eta(3-\eta^2)}{64A} + \alpha^6\left\{\frac{25\kappa^2\eta(\eta^4-1)}{32\pi^2 A} + \right.$$
$$\left. + \frac{25\kappa^2\eta}{3072\,A}(-261\eta^4 + 230\eta^2 - 45)\right\} ,$$

$$S_{\phi}{}^{\tau} = \frac{\alpha^3 5\kappa\eta(1-\eta^2)}{8\pi} - \frac{\alpha^5 25\kappa^2\eta^3(1-\eta^2)}{64} ,$$

$$S_{\tau}{}^{\phi} = -\frac{5\kappa\eta\alpha^3}{8\pi A^2} + \frac{\alpha^5 5\kappa^2\eta}{128A^2}(13\eta^2+9) ,$$

$$S_{\eta}{}^{\eta} = S_{\eta}{}^{\phi} = S_{\phi}{}^{\eta} = S_{\tau}{}^{\eta} = S_{\eta}{}^{\tau} = 0 . \qquad (3.3.2)$$

We note that $S_i^{\ j} = 0$ when $\eta = 0$ i.e. on the edge of the disk. Also the angular momentum density $S_\phi^{\ \tau}$ goes to zero at the center $(\eta=1)$.

IV. Physical Properties of the Disk.

The metric and the energy tensor can be used to calculate several physical parameters which characterize the disk.

The gravitational mass m of the disk can be calculated from the following integral [7]

$$m = - \int_{\Sigma_2} (-g^{\tau\tau})^{-\frac{1}{2}} (S_\tau^{\ \tau} - S_\phi^{\ \phi} - S_\eta^{\ \eta}) \, d\Sigma_2 \tag{3.4.1}$$

where the integration is over the surface of the disk. From (1.1.9) and (2.2.5) we have

$$(-g^{\tau\tau})^{-\frac{1}{2}} d\Sigma_2 = A^2 \eta \, e^{(\nu-\lambda)} \, d\eta \, d\phi \quad . \tag{3.4.2}$$

Hence, using the appropriate expressions for the $S_i^{\ j}$ we conclude that

$$m = \frac{\alpha^2 \kappa A}{2} \tag{3.4.3}$$

a result which is correct up to terms of order α^6 .

Similarly the angular momentum of the disk is

$$J = \int_{\sum 2} (-g^{\tau\tau})^{-\frac{1}{2}} S_{\phi}{}^{\tau} \, d\textstyle\sum_2 \qquad (3.4.4)$$

from which it follows that

$$J = \frac{\alpha^3 \kappa A^2}{6} \quad . \qquad (3.4.5)$$

The surface rest mass density σ was defined by equation (2.1.10) i.e.

$$S_a{}^b u^a = -\sigma u^b \quad , \qquad (3.4.6)$$

$$u^a u_a = -1 \quad .$$

Using the values of $S_i{}^j$ given in (3.3.2) leads to the result

$$\sigma = \frac{\alpha^2 3\kappa\eta}{4\pi A} - \frac{\alpha^4 9\kappa^2\eta}{64\ A}(5-3\eta^2) + \alpha^6 \{ - \frac{25\kappa^2\eta(\eta^4-1)}{32\pi^2\ A} +$$

$$+ \frac{25\kappa^2\eta}{3072A}(125\eta^4 - 190\eta^2 + 141) \} \quad . \qquad (3.4.7)$$

From figure 1 we observe that the surface mass density decreases from its linearized value with the maximum decrease at approximately $\eta = \frac{3}{4}$. Bardeen [8], in his discussion of a uniformly rotating disk, has performed numerical calculations which show that for large expansion parameters, the proper surface density has a maximum near the edge of the disk.

The angular velocity (2.2.15) as seen by an observer at infinity is given by

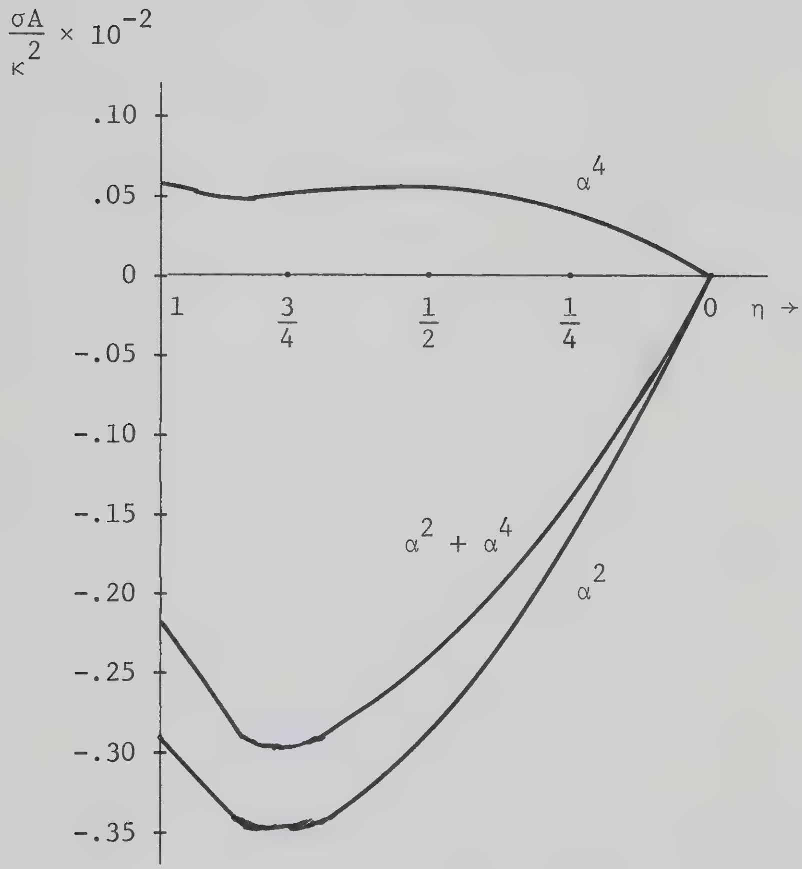

FIGURE I

Corrections to Newtonian $\frac{\sigma A}{\kappa^2}$ $(\frac{81\alpha^2\eta}{200})$ for $\alpha^2 = \frac{1}{10}$.

$$\omega = \frac{5\alpha}{6A} - \frac{5\kappa\pi\,\eta^2\alpha^3}{6A} \quad . \tag{3.4.8}$$

It is evident that ω attains its maximum at the rim of the disk and its minimum at the center.

The surface energy density σ can be used to obtain the rest mass [9]

$$m_o = \int_{\Sigma_2} \sigma\, u^\tau\, (-g^{\tau\tau})^{-\frac{1}{2}}\, d\Sigma_2 \tag{3.4.9}$$

where u^τ can be obtained from the relations

$$u^a u_a = -1 \quad ,$$

$$u^\phi = \omega\, u^\tau \quad .$$

which give

$$u^\tau = [-(g_{\tau\tau} + 2\omega\, g_{\tau\phi} + \omega^2 g_{\phi\phi})]^{-\frac{1}{2}} \quad . \tag{3.4.10}$$

Making use of (3.4.2), (3.4.7), (3.4.10), (3.4.8) and (3.2.4) the rest mass (3.4.9) is

$$m_o = \frac{\alpha^2\kappa A}{2} + \frac{\alpha^4 3\kappa^2\pi A}{80} + \frac{\alpha^6 325\pi\kappa^2 A}{2{,}688} \quad . \tag{3.4.11}$$

The gravitational mass m and the rest mass m_o can be used to calculate the binding energy E_b. The negative of the binding energy

is defined as the energy required to remove matter from the disk to infinity, i.e. the energy required to overcome gravitational forces and rotation, thus

$$-E_b = m - m_o \quad . \tag{3.4.12}$$

From (3.4.3) and (3.4.10) we obtain

$$E_b = \frac{\alpha^4 3\kappa^2 \pi A}{80} + \frac{\alpha^6 325\pi\kappa^2 A}{2,688} \; . \tag{3.4.13}$$

Thus energy is released when the disk is formed.

Using (3.4.11) and (3.4.3) we can determine the fractional binding energy

$$\frac{E_b}{m_o} = \alpha^2 \frac{3\kappa\pi}{40} + \alpha^4 \frac{311\pi\,\kappa}{1,344} \quad . \tag{3.4.14}$$

According to Bardeen's numerical computations the fractional binding energy is a positive monotonically increasing function of the expansion parameter.

CHAPTER IV

Applications of the Metric

I. Geometry of the Disk.

Since the gravitational potentials g_{ij} are assumed to be continuous functions, it follows that the exterior field uniquely determines the metric on the disk.

In special relativity an element, $d\ell$ of spatial distance is defined to be the interval between two infinitismally separated events occurring at the same time. However in general relativity proper time is a function of position. Hence, it is necessary to make a correction to the t = constant form (see Landau and Lifshitz [10])

$$d\ell^2 = \gamma_{ab}\, dx^a\, dx^b \tag{4.1.1}$$

where

$$\gamma_{ab} = g_{ab} - \frac{g_{ta} g_{tb}}{g_{tt}} .$$

For the disk $(\epsilon=0)$ it follows from (1.2.30) that

$$d\ell^2 = \frac{A^2 \eta^2 e^{2(\nu-\lambda)}}{1-\eta^2}\, d\eta^2 + A^2(1-\eta^2) e^{-2\lambda}\, d\phi^2 . \tag{4.1.2}$$

The coordinate radius A of the disk has no invariant meaning. To obtain a proper radius to the disk we note that the above metric is

that of flat space in oblate spheroidal coordinates for η approaching ± 1 where the radius of the disk is given by

$$A' = A \exp(-\lambda_{\eta=\pm 1}) \quad . \tag{4.1.3}$$

Thus we rewrite (4.1.2) as

$$d\ell^2 = \frac{(Ae^{-\lambda_c})^2\eta^2}{1-\eta^2}\{e^{2(\nu-\lambda+\lambda_c)}\}\,d\eta^2 \quad +$$

$$+ (A\,e^{-\lambda_c})^2(1-\eta^2)\,\{e^{2(\lambda_c-\lambda)}\}\,d\phi^2 \tag{4.1.4}$$

where λ_c is the λ - function evaluated at the center of the disk ($\eta = \pm 1$) . The bracketed quantities above give the deviation of the disks geometry from the flat space determined by elementary flatness.

Using (3.2.3) we can write (4.1.4) as

$$d\ell^2 = \frac{(Ae^{-\lambda_c})^2\eta^2}{1-\eta^2}\{1 + 2\alpha^2(\lambda_{oc}-\lambda_o) + 2\alpha^4(\nu_1+\lambda_o{}^2+\lambda_{oc}{}^2-2\lambda_o\lambda_{oc}) \quad +$$

$$+ \quad \alpha^6(2\nu_2 - 2\lambda_1 - \frac{4}{3}\lambda_o{}^3 + 4\nu_1\lambda_{oc} - 4\nu_1\lambda_o + 4\lambda_o{}^2\lambda_{oc} + 2\lambda_{1c} \quad -$$

$$- 4\lambda_o\lambda_{oc}{}^2 + \frac{4}{3}\lambda_{oc}{}^3)\}\,d\eta^2 \;+ (Ae^{-\lambda_c})^2\,(1-\eta^2)\,\{1 + 2\alpha^2(\lambda_{oc}-\lambda_o) \quad +$$

$$+ 2\alpha^4(\lambda_{oc}{}^2 - 2\lambda_{oc}\lambda_o + \lambda_o{}^2) + \alpha^6(-2\lambda_1 - \frac{4}{3}\lambda_o{}^3 + 2\lambda_{1c} + 4\lambda_1{}^2\lambda_{oc} \quad -$$

$$- 4\lambda_o\lambda_{oc}{}^2 + \frac{4}{3}\lambda_{oc}{}^3)\}\,d\phi^2 \quad . \tag{4.1.5}$$

In the above expression rotational effects are observed in the α^6 approximation in the form of the functions λ_1 and ν_2 whose source is ψ_o .

The proper circumferential radius of a circle about the axis of symmetry for constant η is

$$r = \frac{C}{2\pi} = \frac{1}{2\pi} \int_{\text{const } \eta,\tau} (d\ell^2)^{\frac{1}{2}} = A\, e^{-\lambda_c} (1-\eta^2)^{\frac{1}{2}} \{1 + \alpha^2(-\lambda_o + \lambda_{oc}) +$$

$$+ \alpha^4(\tfrac{1}{2}\lambda_o^2 - \lambda_o\lambda_{oc} + \tfrac{1}{2}\lambda_{oc}^2) + \alpha^6(-\lambda_1 - \frac{\lambda_o^3}{6} + \tfrac{1}{2}\lambda_o^2\lambda_{oc} - \tfrac{1}{2}\lambda_o\lambda_{oc}^2 +$$

$$+ \lambda_{1c} + \frac{\lambda_{oc}^3}{6})\} \quad . \tag{4.1.6}$$

Using (B.1) one obtains explicitly

$$r = A\, e^{-\lambda_c} (1-\eta^2)^{\frac{1}{2}} \{1 + \frac{\alpha^2 3\kappa\pi}{16}(\eta^2-1) + \frac{\alpha^4 9\kappa^2\pi^2}{512}(\eta^2-1)^2 +$$

$$+ \alpha^6 [\frac{25\kappa^2}{36}(\eta^6-1) + \frac{9\kappa^3\pi^3}{8{,}192}(\eta^2-1)^3]\} + 0(\alpha^8) \quad . \tag{4.1.7}$$

Thus the proper circumferential radius will decrease from its flat space value of $A\, e^{-\lambda_c}(1-\eta^2)^{1/2}$.

II. Asymptotic Properties.

We would like to examine the disk metric for large ε. Since we have postulated asymptotic flatness it is necessary that $\varepsilon = \infty$ reduces the disk metric to that of Minkowski space-time. If we take into account first order corrections from flatness (α^2-terms) we expect a certain simularity to the asymptotic form of the Schwartzschild metric. If we consider rotational terms (α^3-terms) the far field of the disk should have the form of the linearized metric of a rotating ball of fluid.

In order to substantiate the above expected properties we will compare the disk metric for large ε to the asymptotic form of the Kerr metric [11] which is the only known stationary exact solution which obeys asymptotic flatness.

The Kerr metric can be derived by the Ernst function [4] technique and has the following form in prolate spheroidal coordinates

$$ds^2 = \frac{x^2 + 2xm + m^2 + a^2y^2}{x^2 - y^2} \{(x^2-y^2) [\frac{dx^2}{(x^2-1)} + \frac{dy^2}{(1-y^2)}]\} +$$

$$+ [x^2 + 2xm + m^2 + a^2 + \frac{2m(x+m)\ a^2(1-y^2)}{x^2+2xm+m^2+a^2y^2}]\ (1-y^2)\ d\phi^2 +$$

$$+ \frac{4am(x+m)(1-y^2)}{x^2+2xm+m^2+a^2y^2}\ d\phi\ dt - [1 - \frac{2m(x+m)}{x^2+2xm+m^2+a^2y^2}]\ dt^2 ,$$

where prolate spheroidal coordinates (x,y) are defined by

$$\rho = A(x^2-1)^{\frac{1}{2}} (1-y^2)^{\frac{1}{2}} \tag{4.2.2}$$

$$z = A\, xy$$

and we have set $A = 1$ in (4.2.1).

We note that the relationship between polar coordinates, oblate spheroidal coordinates and prolate spheroidal coordinates is given by

$$\left(\frac{r}{A}\right)^2 = 1 + \varepsilon^2 - n^2 \quad ,$$

$$\left(\frac{r}{A}\right) \cos\theta = \varepsilon\, n \quad ,$$

$$\left(\frac{r}{A}\right)^2 = x^2 - (1-y^2) \quad , \tag{4.2.3}$$

$$\left(\frac{r}{A}\right) \cos\theta = + xy \quad .$$

Comparing (4.2.1) and (1.1.9) we obtain

$$e^{2\lambda} = \frac{x^2 - m^2 + a^2y^2}{x^2 + 2xm + m^2 + a^2y^2} \quad ,$$

$$e^{-2\lambda} = \frac{x^2 + 2xm + m^2 + a^2y^2}{x^2 - m^2 + a^2y^2} \quad ,$$

$$\psi = \frac{2am(1-y^2)(x+m)}{x^2 - m^2 + a^2y^2} \quad , \tag{4.2.4}$$

$$e^{2\nu} = \frac{x^2 - m^2 + a^2y^2}{x^2 - y^2} \quad .$$

Expanding the above functions for large x and rewriting the expansions in terms of (r,θ) gives the following results:

$$e^{2\lambda} \sim 1 - \frac{2m}{r} + \frac{2m^2}{r^2} + \frac{1}{r^3}\{m \sin^2\theta - 2m^3 + 2m\, a^2 \cos^2\theta\} \quad ,$$

$$e^{-2\lambda} \sim 1 + \frac{2m}{r} + \frac{2m^2}{r^2} - \frac{1}{r^3}\{m \sin^2\theta - 2m^3 + 2m\, a^2 \cos^2\theta\} \quad , \qquad (4.2.5)$$

$$\psi \sim \frac{2am \sin^2\theta}{r} + \frac{2am^2 \sin^2\theta}{r^2} - \frac{2am \sin^2\theta}{r^3}\{\frac{1}{2}\sin^2\theta - \cos^2\theta +$$

$$+ a^2\cos^2\theta - m^2\} \quad ,$$

$$e^{2\nu} \sim 1 + \frac{(a^2+1)\cos^2\theta - m^2}{r^2} + \cdots$$

where the above expressions are good to order $(\frac{1}{r^3})$.

The asymptotic expansion of the disk metric to order $(\frac{1}{r^3})$ and order α^6 with $A = 1$ has the form:

$$e^{2\lambda} \sim 1 - \frac{k\alpha^2}{r} + \frac{\alpha^4 k^2}{2r^2} - \frac{1}{r^3}\{\frac{k^2}{5}(\cos^2 -2) + \frac{k\alpha^2}{2}\sin^2\theta + \frac{\alpha^6 k^3}{6}\} ,$$

$$e^{-2\lambda} \sim 1 - \frac{k\alpha^2}{r} + \frac{\alpha^4 k^2}{2r^2} - \frac{1}{r^3}\{\frac{k\alpha^2}{5}(\cos^2\theta - 2) + \frac{k\alpha^2}{2}\sin^2\theta + \frac{\alpha^6 k^3}{6}\} \quad ,$$

$$\psi \sim -\frac{\alpha^3 k \sin^2\theta}{3r} - \frac{\alpha^5 k^2 \sin^2\theta}{6r^2} - \frac{\alpha^3 k \sin^2\theta}{r^3}\{\frac{\sin^2\theta}{6} - \frac{1}{3}\cos^2\theta +$$

$$+ \frac{\cos^2\theta}{7} - \frac{2}{21}\} \quad . \qquad (4.2.6)$$

Comparing the Kerr metric (4.2.5) to the disk metric (4.2.6) we observe that

$$m = \frac{k\alpha^2}{2} \tag{4.2.7}$$

must be the graviational mass of the disk, which agrees with (3.4.3). The parameter a of the Kerr metric must correspond to

$$a = -\frac{\alpha}{3} \quad . \tag{4.2.8}$$

This is the value of a which could be obtained by interpreting the quantity $(-ma)$ of the Kerr metric as the total angular momentum of the disk given by (3.4.5).

Rewriting the disk metric in terms of the above two parameters we obtain

$$\begin{aligned} e^{2\lambda} &\sim 1 - \frac{2m}{r} + \frac{2m^2}{r^2} - \frac{1}{r^3}\left\{\frac{2m}{5}(\cos^2\theta - 2) + m\sin^2\theta + \frac{4m^3}{3}\right\} , \\ e^{-2\lambda} &\sim 1 + \frac{2m}{r} + \frac{2m^2}{r^2} + \frac{1}{r}\left\{\frac{2m}{5}(\cos^2\theta - 2) + m\sin^2\theta + \frac{4m^3}{3}\right\} , \\ \psi &\sim \frac{2am\sin^2\theta}{r} + \frac{2am^2\sin^2\theta}{r^2} + \frac{2am\sin^2\theta}{r^3}\left\{\frac{\sin^2\theta}{2} - \cos^2\theta + \right. \\ &\quad \left. + \frac{3\cos^2\theta}{7} - \frac{2}{7}\right\} . \end{aligned} \tag{4.2.9}$$

The Schwartzchild metric can be obtained from the Kerr metric by letting the parameter a go to zero. Comparing (4.2.9) and (4.2.5)

with $a = 0$ shows the disk deviates from the Schwartzchild metric in the $(\frac{1}{r^3})$ terms with the minimum deviation obtained for an observer along the axis of symmetry. If we consider the effects of rotation $(a \neq 0)$ the two metrics coincide to order $(\frac{1}{r^2})$. Since the $(\frac{1}{r})$ terms of the Kerr metric in Schwartzchild-like coordinates are the same as the linearized metric of a rotating ball of fluid [12] we infer that the disk metric is asymptotic to the linearized metric of a rotating ball of fluid.

For a further discussion of the disk metric and the Kerr metric in the strong field limit we refer the reader to Bardeen [8].

III. The Gravitational Red Shift.

In this section we obtain an expression for the change in the frequency of a photon due to the gravitational field of the disk.

Consider a radiating atom at the center of the disk $(0,1)$ and an observer at (ε,η). The proper time intervals at the center and at (ε,η) are denoted respectively by

$$\Delta\tau_c = (-g_{tt_c})^{\frac{1}{2}} \Delta t \quad ,$$
$$\Delta\tau = (-g_{tt})^{\frac{1}{2}} \Delta t \quad . \tag{4.3.1}$$

If an atom at the center (in proper time $\Delta\tau_c$) emits n-waves at frequency ν_o then we have

$$n = \nu_o \, \Delta\tau_c \quad . \qquad (4.3.2)$$

At the observation point the receiver would observe the same n-waves in some proper time interval $\Delta\tau$. Evidently the observed frequency ν and the proper time $\Delta\tau$ would be related by

$$n = \nu \, \Delta\tau \quad . \qquad (4.3.3)$$

Hence,

$$\nu = \nu_o \frac{\Delta\tau_c}{\Delta\tau} = \nu_o \left(\frac{g_{tt}}{g_{tt}}\right)^{\frac{1}{2}} \quad . \qquad (4.3.4)$$

The red shift is given by

$$\Delta \nu = \nu_o \left\{\left(\frac{g_{tt_c}}{g_{tt}}\right)^{\frac{1}{2}} - 1\right\} \quad . \qquad (4.3.5)$$

From (3.2.4) we have

$$g_{tt} = - \{1 + 2\alpha^2\lambda_o + 2\alpha^4\lambda_o^2 + \alpha^6(2\lambda_1 + \frac{4}{3}\lambda_o^3)\} \ . \qquad (4.3.6)$$

Using (4.3.6), (4.3.5) and the appropriate expressions in appendix B we obtain (correct to order α^6)

$$\begin{aligned} \frac{\Delta\nu}{\nu_o} = {} & \alpha^2 \left(- \frac{3k\pi}{8} - \lambda_o\right) + \alpha^4\left(\frac{1}{2} \lambda_o^2 + \frac{3k\pi}{8} \lambda_o + \frac{9k^2\pi^2}{128}\right) + \\ & + \alpha^6 \left(-\lambda_1 - \frac{25k^2}{36} - \frac{\lambda_o^3}{6} - \frac{3k\pi}{16} \lambda_o^2 - \frac{9k^2\pi^2}{128} \lambda_o - \frac{27k^3\pi^3}{3072}\right) \ . \end{aligned} \qquad (4.3.7)$$

Evidently the maximum red shift occurs for an observer at infinity where $\lambda_o = \lambda_1 = 0$. In this case (4.3.7) reduces to

$$\frac{\Delta\nu}{\nu_o}\bigg|_{\varepsilon=\infty} = -\alpha^2 \frac{3k\pi}{8} + \alpha^4 \frac{9k^2\pi^2}{128} + \alpha^6 \left(-\frac{25k^2}{36} - \frac{27k^3\pi^3}{3072}\right) \quad . \qquad (4.3.8)$$

On the disk $(\varepsilon=0)$ the red shift (now a function of η) is given by

$$\frac{\Delta\nu}{\nu_o}\bigg|_{\varepsilon=0} = \alpha^2 \left[\frac{3k\pi}{16}(n^2-1)\right] + \alpha^4 \left[\frac{9k^2\pi^2}{512}(n^2-1)^2\right] +$$

$$+ \alpha^6 \left[\frac{27k^3\pi^3}{24,576}(n^2-1)^3 + \frac{25k^2}{36}(n^6-1)\right] \quad . \qquad (4.3.9)$$

From the above expression it is observed that to order α^4 the red shift is identical to that of a static disk. The rotational term in the α^6 approximation decreases the frequency of light emitted at the center of the disk.

The maximum red shift on the disk is obtained at the edge. Setting $\eta = 0$ we obtain

$$\frac{\Delta\nu}{\nu_o}\bigg|_{\substack{\eta=0\\ \epsilon=0}} = -\frac{\alpha^2 3k\pi}{16} + \alpha^4 \frac{9k^2\pi^2}{512} + \alpha^6 \left[-\frac{27k^3\pi^3}{24,576} - \frac{25k^2}{36}\right] \quad . \qquad (4.3.10)$$

To obtain the red shift of a photon emitted at the edge of the disk and received at infinity we subtract (4.3.10) from (4.3.8) which gives

$$\frac{\Delta\nu}{\nu_o} = -\alpha^2 \frac{3k\pi}{16} + \alpha^4 \frac{27k^2\pi^2}{512} - \alpha^6 \frac{189\ k^3\pi^3}{24,576} \quad . \qquad (4.3.11)$$

If we only consider α^2 - terms, half of the total energy loss of the photon in the $z = 0$ plane is obtained when the photon reaches the edge of the disk the other half is lost in leaving the external gravitational field of the disk. We also observe that rotation doesn't effect the red shift of a photon emitted at the rim of the disk in the order of approximation we have considered.

The red shift of a photon emitted at the surface of a spherical mass distribution of mass m and radius A is given by

$$\frac{\Delta\nu}{\nu_o} = -\frac{m}{A} \quad . \tag{4.3.12}$$

Using (3.4.3) we can write (4.3.11) as

$$\frac{\Delta\nu}{\nu_o} = -\frac{3\pi}{8}\frac{m}{A} + \frac{\pi^2}{128}\left(\frac{m}{A}\right)^2 - \frac{189\ \pi^3}{3,072}\left(\frac{m}{A}\right)^3 \quad . \tag{4.3.13}$$

Thus the energy loss will be slightly greater for a photon leaving a disk in the $z = 0$ plane than a spherical object.

IV. Lense-Thirring Effect.

In this section we consider the effect of the $g_{t\phi}$ component of the metric tensor which gives rise to the so-called "dragging of the inertial frames" of an observer.

Hartle and Sharp [17] have shown that the axially symmetric stationary metric can be written in the form

$$ds^2 = - H^2 dt^2 + Q^2 dr^2 + r^2 K^2 [d\theta^2 + \sin^2\theta\,(d\phi - L\,dt)^2] \qquad (4.5.1)$$

where H , Q , K and L depend on (r,θ). By an argument analogous to that given by Hartle [17] the function L is the angular velocity of an observer who falls from infinity to the point (r,θ) . Thus L is the rate of rotation of the inertial frame with respect to flat space.

If (1.1.9) and (4.5.1) are compared it follows that

$$L = - \frac{\psi\, e^{2\lambda}}{r^2 \sin^2\theta\, e^{-2\lambda} - e^{2\lambda}\psi^2} \qquad (4.5.2)$$

or to order α^3

$$L = - \frac{\psi}{A^2(1+\varepsilon^2)(1-\eta^2)} \quad . \qquad (4.5.3)$$

On the surface of the disk (4.5.3) becomes

$$L = \alpha^3 \frac{5k\,\pi\,(1+3\eta^2)}{32A} \quad . \qquad (4.5.4)$$

We observe the rotation is greatest at the center of the disk.

Now the ω of (3.4.8) is the angular velocity of the fluid as seen by an observer at infinity.

Hence,

$$\tilde{\omega} = \omega - L \qquad (4.5.5)$$

is the rate of rotation of the fluid with respect to a local inertial frame as seen by an observer at infinity.

From (4.5.5), (4.5.4) and (3.4.8) one concludes that for points on the disk

$$\tilde{\omega} = \frac{5\alpha}{6A} - \frac{\alpha^3 5k\ \pi}{96A}\,(3+25\eta^2) \quad . \tag{4.5.6}$$

Thus the rate of rotation of the fluid with respect to an inertial frame is less than with respect to an observer at infinity. The decrease is observed to be maximum at the center and minimum at the edge.

CHAPTER V

Summary and Conclusions

In this final chapter we would like to summarize the results that have been obtained.

The first chapter was concerned with determining the linearized field of a uniformly rotating, infinitismally thin disk. The choice of uniform rotation was governed by two considerations (i) the work of Hunter [13] (a Newtonian treatment of infinitismally thin disks) has shown that the simplest possible mass distribution $\sigma = 2\,\sigma_o\eta$ leads to uniform rotation, (ii) the possible use of a thin uniformly rotating disk as a model for a galaxy (rotation is observed to be approximately uniform in certain galactic systems [14]).

Concerning the astrophysical implications of a thin rotating disk it should also be emphasised that Hunter has also shown that an infinitismally thin Newtonian disk is unstable. Since general relativity enhances instabilities we would not expect the relativistic disk to be stable.

There are two possible generalizations which may improve the stability of the disk, namely, (i) differential rotation, (ii) assume the disk has thickness. Differential rotation can be obtained for a disk by assuming σ is a polynomial in η . The techniques used to determine the linearized metric could then be carried out. However, the work

involved would be considerable. For a further discussion of stability we refer the reader to Bardeen [8].

The second chapter contains explicit expressions for the energy tensor. It is shown that the proper surface density and angular velocity (correct to order α^3) correspond to their appropriate Newtonian values. From equation (2.2.12) it follows that rotation does not effect the Newtonian surface density until one considers terms of order α^4.

In the third chapter the gravitational field was determined to order α^6. Integral expressions for the functions which determine the gravitational field were obtained correct to order α^8. However, the integrals are so unwieldy that these results are not included.

It was also noted that the angular velocity ω, (3.4.8) is no longer constant but becomes a function of η. The matter rotates faster at the edge than at the center, which is opposite to what one would want for a realistic galaxy. In order to obtain a disk which is uniformly rotating one would have to choose a different initial mass distribution.

We also observed that the calculated binding energy and fractional binding energy are positive which is in agreement with that of a incompressible fluid sphere [15].

Chapter IV contains a discussion of certain properties of the disk metric. In particular we considered such effects as the gravitational red shift, asymptotic properties, and the dragging of inertial frames.

When the metric of the disk was compared to the Kerr metric the observed deviations were in the terms of order $(\frac{1}{r^3})$. The deviation was minimal for an observer on the axis of symmetry.

The red shift which we considered was due only to the gravitational field. This is not the red shift which is physically observed since there is an additional shift in the frequency of light due to the rotation of the disk. The effects of the rotational graviational field on the photons energy loss appeared only in the α^6 approximation causing a greater red shift than in the static case.

The effects of rotation were also observed in the so-called "dragging of the inertial frames". This effect was calculated to order α^3 . It was concluded than an inertial observer sees the fluid rotate faster at the edge than at the center.

By determining the gravitational field of a rotating disk we have also obtained the field to a static disk. To see this we note from equations (1.1.10), (1.1.11) and (1.1.12) that $\psi = 0$ gives equation (1.1.20) which we have solved. Thus, although we can't let ψ be zero in our above solution the results obtained for $\psi = 0$ is the same as the results which one would obtain by solving the static field equations.

In conclusion we would like to comment on the relative strengths of the above effects. The expansion parameter is given by $\alpha^2 = \frac{27\pi m}{25A}$. If one considers the galaxy NGC 2782 which obeys a linear rotation law [14] and has a mass and radius given by

$$m = 17 \times 10^{10} \; m_{\odot} \qquad\qquad R = 1.3 \text{ k.p.c}$$

it follows that

$$\alpha^2 = 2.2 \times 10^{-5} \quad .$$

We thus observe that for a normal galaxy our expansion parameter is indeed less than one. One might therefore expect that the description of such galaxys in terms of the linearized solution to be quite good. Of course for the more interesting case of strong fields (i.e. where $M \sim A$) the above analysis would be inadequate.

Finally we refer the reader to Cameron [16] for a detailed discussion of the astrophysical applications of rotating disks and to Bardeen [8] for a complete numerical treatment of the disk problem including the strong field limit.

APPENDIX A

Integration of (1.1.23) and (1.1.24) using λ_o and ψ_o given by (1.2.13) and (1.2.28) gives the following result for ν_1 and ν_2 .

$$\nu_1 = \frac{\kappa^2(1+\varepsilon^2)}{4} \{[\frac{\eta^4}{4} - \frac{\varepsilon^2\eta^2}{2} + \frac{\varepsilon^4}{2} \log(\varepsilon^2+\eta^2) - \frac{\varepsilon^4}{2} \log(\varepsilon^2+1) +$$

$$+ \frac{\varepsilon^2}{2} - \frac{1}{4}] [\frac{9}{4} (Q_2')^2(1+\varepsilon^2) + 9(Q_2)^2 - 9\varepsilon Q_2Q_2'] +$$

$$+ [\frac{\eta^2}{2} - \frac{\varepsilon^2}{2} \log (\varepsilon^2+\eta^2) + \frac{\varepsilon^2}{2} \log (\varepsilon^2+1) - \frac{\eta^2}{2}] [-(1+\varepsilon^2) \times$$

$$\times (\frac{3}{2} (Q_2')^2 + 3Q_o'Q_2') - 9(Q_2)^2 + 6\varepsilon Q_o'Q_2 + 12\varepsilon Q_2Q_2'] +$$

$$+ [\frac{1}{2} \log (\varepsilon^2+\eta^2) - \frac{1}{2} \log (\varepsilon^2+1)] [(1+\varepsilon^2)((Q_o')^2 + \frac{(Q_2')^2}{4} +$$

$$+ Q_o'Q_2') - 6\varepsilon (Q_o'Q_2 + \frac{1}{2} Q_2Q_2')]\} \tag{A.1}$$

$$\nu_2 = \frac{\kappa^2(1+\varepsilon^2)}{4} \{[\frac{\eta^6}{6} - \frac{\varepsilon^2\eta^4}{4} + \frac{\varepsilon^4\eta^2}{2} - \frac{\varepsilon^6}{2} \log (\varepsilon^2+\eta^2) - \frac{1}{6} + \frac{\varepsilon^2}{4} -$$

$$- \frac{\varepsilon^4}{2} + \frac{\varepsilon^6}{2} \log (\varepsilon^2+1)] [\frac{25}{4} (Q_3')^2 (1 + \frac{1}{4} \varepsilon^2) + \frac{25}{64} (1+\varepsilon^2)^2 \times$$

$$\times (Q_3'')^2 - \frac{25}{16} \varepsilon(1+\varepsilon^2) Q_3'Q_3''] + [\frac{\eta^4}{4} - \frac{\varepsilon^2\eta^2}{2} + \frac{\varepsilon^4}{2} \log (\varepsilon^2+\eta^2)$$

$$- \frac{1}{4} + \frac{\varepsilon^2}{2} - \frac{\varepsilon^4}{2} \log (\varepsilon^2+1)] [(Q_3')^2 \{- \frac{15}{2} (1+\varepsilon^2) + \frac{145}{16} \varepsilon^2\} -$$

$$- 5 Q_3'Q_1' - \frac{35}{64} (1+\varepsilon^2)^2 (Q_3'')^2 - \frac{5}{8} (1+\varepsilon^2)^2 Q_1''Q_3'' +$$

$$+ \frac{55}{16} \varepsilon(1+\varepsilon^2) Q_3''Q_3' + \frac{5}{4} \varepsilon(1+\varepsilon^2) Q_1''Q_3'] + [\frac{\eta^2}{2} - \frac{\varepsilon^2}{2} \log (\varepsilon^2+\eta^2) -$$

$$- \frac{1}{2} + \frac{\varepsilon^2}{2} \log (\varepsilon^2+1)] [\frac{9}{16} (Q_3')^2 (4-5\varepsilon^2) + Q_3'Q_1' (3-5\varepsilon^2) +$$

$$+ (Q_1')^2 + \frac{11}{64} (1+\varepsilon^2)^2 (Q_3'')^2 + \frac{1}{4} (1+\varepsilon^2)^2 (Q_1'')^2 + \frac{3}{4} (1+\varepsilon^2)^2 Q_1''Q_3'' -$$

$$- \varepsilon(1+\varepsilon^2)\{ \frac{35}{16} Q_3'Q_3'' - \frac{5}{2} Q_3'Q_1''\}] + [\frac{1}{2} \log (\varepsilon^2+\eta^2) - \frac{1}{2} \log (\varepsilon^2+1)] \times$$

$$\times [\frac{11}{16} (Q_3')^2 \varepsilon^2 + (Q_1')^2 \varepsilon^2 + 3 Q_1'Q_3' \varepsilon^2 - \frac{1}{4} (1+\varepsilon^2)^2(Q_1'')^2 -$$

$$- \frac{1}{64} (Q_3'')^2 (1+\varepsilon^2)^2 - \frac{1}{8} Q_1''Q_3'' (1+\varepsilon^2)^2 + \varepsilon(1+\varepsilon^2)\{ \frac{5}{4} Q_1''Q_3' +$$

$$+ \frac{5}{16} Q_3'Q_3''\}]\} \quad . \qquad (A.2)$$

The above expression satisfies both asymptotic flatness and elementary flatness ($\nu \to 0$ as $\eta \to \pm 1$).

APPENDIX B

We list the functions λ_o , λ_1 , ψ_o , ψ_1 , ν_1 , ν_2 and their derivatives with respect to ε evaluated on the disk $(\epsilon=0)$.

$$
\begin{aligned}
\lambda_o &= -\frac{3\kappa\pi}{16}(1+\eta^2) \quad , \\
\lambda_{o\varepsilon} &= \frac{3\kappa\eta^2}{2} \quad , \\
\psi_o &= -\frac{5\kappa A\pi(1-\eta^2)(3\eta^2+1)}{32} \quad , \\
\psi_{o\varepsilon} &= 5\kappa\, A\, \eta^2(1-\eta^2) \quad , \\
\psi_1 &= -\frac{5\kappa^2\, A(1-\eta^2)(\eta^4+\eta^2+1)}{6} - \frac{15\kappa^2\pi^2\, A(1-\eta^2)(1+\eta^2)^2}{128} \quad , \qquad (B.1) \\
\psi_{1\varepsilon} &= \frac{5\kappa^2\, A\pi\, \eta^2(1-\eta^2)(3+\eta^2)}{8} \quad , \\
\lambda_1 &= -\frac{25\kappa^2\eta^6}{36} \quad , \\
\lambda_{1\varepsilon} &= \frac{25\kappa^2\pi\, \eta^4(3\eta^2-1)}{48} \quad , \\
\nu_1 &= \frac{9\kappa^2}{16}(\eta^4-1) + \frac{9\kappa^2\pi^2}{256}(\eta^2-1)^2 \quad , \\
\nu_{1\varepsilon} &= \frac{9\kappa^2\pi\eta^2}{8}(1-\eta^2) \quad , \\
\nu_2 &= -\frac{25\kappa^2\pi^2}{512}(1-\eta^2)(1+3\eta^4) + \frac{25\kappa^2}{48}(1-\eta^2)(1+\eta^2-2\eta^4) \quad , \\
\nu_{2\varepsilon} &= \frac{25\kappa^2\, \pi\, \eta^2(1-\eta^2)(3\eta^2-1)}{16} \quad .
\end{aligned}
$$

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